ON A PROBLEM OF MIHLIN

BY

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1. Let $x=(\xi_1, \xi_2, \cdots, \xi_k)$, $y=(\eta_1, \eta_2, \cdots, \eta_k)$, $z=(\zeta_1, \zeta_2, \cdots, \zeta_k)$, $\cdots$ denote points of the k-dimensional Euclidean space E^k. Here $k \geqq 1$ but only the case $k \geqq 2$ will be of interest. The space may also be treated as a vector space by identifying x with the vector joining the origin $0=(0, \cdots, 0)$ with the point x. The rules for addition of vectors and for multiplying them by scalars are the usual ones, and the norm is defined by the formula

$$|x| = (\xi_1^2 + \xi_2^2 + \cdots + \xi_k^2)^{1/2}.$$

By $x'=(\xi_1', \xi_2', \cdots, \xi_k')$ we shall systematically denote the point of intersection of the ray $0x$ $(x \neq 0)$ with the unit sphere $\Sigma=\Sigma_{k-1}$ defined by the equation $|x|=1$. Thus,

$$x' = x/|x|, \qquad |x'| = 1.$$

In this note we shall consider the problem of the existence of the integral

$$\int_{E^k} K(x, y)f(y)dy \tag{1.1}$$

where $dy=d\eta_1 d\eta_2 \cdots d\eta_k$, f is a function of the class L^2 over E^k, and $K(x, y)$ is a singular kernel satisfying certain conditions. In general, we shall have

$$K(x, y) = \frac{\Omega(x, z')}{|z|^k} \tag{1.2}$$

where, systematically, $z=x-y$. Thus $K(x, y)$ depends on the point x and on the direction from x to y.

In a special but important case, K may depend on z only. We then have

$$K(x, y) = \frac{\Omega(z')}{|z|^k} \qquad (z = x - y)$$

i.e.

$$K(x, y) = K(x - y) \quad \textit{with} \quad K(x) = \frac{\Omega(x')}{|x|^k}\cdot \tag{1.3}$$

It is well known (see [2]) that if $K(x)$ satisfies certain regularity conditions and the indispensable condition

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$$\int_\Sigma \Omega(x')dx' = 0,$$

and if $f \in L^p$, $p \geqq 1$, then the integral

$$\int_{E^k} K(x-y)f(y)dy \tag{1.4}$$

exists in the principal value sense for almost every x. (By the principal value of the integral (1.4) we mean the limit, for $\epsilon \to 0$, of the integral extended over the exterior of the sphere with center x and radius ϵ.) Moreover, the value $\tilde{f}(x)$ of (1.4) has many properties similar to those of the Hilbert transform

$$\int_{-\infty}^{+\infty} \frac{f(y)dy}{x-y}$$

in E^1.

The purpose of the present note is to prove some results about the more general case (1.2). We fix our kernel $K(x, y)$ once for all and use the notation

$$\tilde{f}_\epsilon(x) = \int_{|z|\geqq\epsilon} K(x, y)f(y)dy \qquad (z = x - y). \tag{1.5}$$

By $\tilde{f}(x)$ we shall mean the limit of $\tilde{f}_\epsilon(x)$ as $\epsilon \to 0$. This limit may be considered pointwise or in some norm. In this note we shall be concerned exclusively with convergence in norm. We shall systematically use the notation

$$\|f\|_p = \left(\int_{E^k} |f(y)|^p dy\right)^{1/p},$$

but in the case $p=2$ we shall simply write $\|f\|$ for $\|f\|_2$.

In what follows, by A with various subscripts we shall mean a constant depending on the kernel K and on the parameters displayed in the subscripts. In particular, by A without any subscript we shall mean constants depending on K at most. The constants need not be the same at every occurrence.

We shall first state the main theorem of this note. Comments and generalizations are postponed to a later section.

THEOREM 1. *Suppose that the kernel K defined by* (1.2) *satisfies for each x the following two conditions*:

$$\int_\Sigma \Omega(x, z')dz' = 0, \tag{1.6}$$

$$\int_\Sigma |\Omega(x, z')|^2 dz' \leqq A, \tag{1.7}$$

with A independent of x. Let $f \in L^2$. Then for each x and $\epsilon > 0$ the integral (1.5) *converges absolutely and the function* $\tilde{f}_\epsilon(x)$ *tends to a limit* $\tilde{f}(x)$ *in norm* L^2. *Moreover,*

$$\|\tilde{f}_\epsilon\| \leqq A\|f\| \qquad (\epsilon > 0), \tag{1.8}$$

$$\|\tilde{f}\| \leqq A\|f\|. \tag{1.9}$$

This theorem was stated as a problem by Mihlin in [6] (see also [5]). He settled the case $k=2$ only, in a somewhat weaker form since he defines $\tilde{f}(x)$ not as a limit of $\tilde{f}_\epsilon(x)$ but as a linear operator which for sufficiently smooth (say, differentiable) functions f is defined directly by the then everywhere convergent integral (1.4) and is subsequently extended by continuity to all functions $f \in L^2$. For $k>2$ he has to replace condition (1.7) by much stronger conditions involving partial derivatives of Ω.

2. Let us fix x and develop $\Omega(x, z')$ into a series of spherical harmonics

$$\Omega(x, z') \sim \sum_{n=1}^{\infty} a_n(x) Y_n(z') \tag{2.1}$$

where $Y_n(z')$ is an (ultra) spherical function of order n, i.e. is the value on Σ of a homogeneous polynomial $P(\zeta_1, \cdots, \zeta_k)$ satisfying Laplace's equation $\Delta P=0$. The development begins with $n=1$ since, on account of (1.6), the term $n=0$ of the development is zero. If $k=2$ we may also write (2.1) in the form

$$\sum_{-\infty}^{+\infty}{}' a_n(x) e^{inx} .$$

We may always normalize the Y_n and assume that

$$\|Y_n\| = \left(\frac{1}{|\Sigma|}\int_\Sigma |Y_n(z')|^2 dz'\right)^{1/2} = 1,$$

$|\Sigma|$ denoting the $(k-1)$-dimensional measure of Σ. No misunderstanding will occur if we use the same notation for the norm in two different cases, those of the whole space E^k and of the sphere Σ.

The functions $Y_n(z')$ form an orthonormal system on Σ and Bessel's inequality combined with (1.6) gives

$$\left(\sum_{n=1}^{\infty} |a_n(x)|^2\right)^{1/2} \leqq A. \tag{2.2}$$

It will be convenient to modify the definition (1.5) by inserting the factor $(2\pi)^{-k/2}$ in the integral. Thus

$$\tilde{f}_\epsilon(x) = (2\pi)^{-k/2}\int_{|z|\geqq\epsilon} f(y)\frac{\Omega(x, z')}{|z|^k} dy \qquad (z = x - y). \tag{2.3}$$

This integral converges absolutely for each x since $f \in L^2$ and $\Omega(x, z')|z|^{-k}$ is, on account of (1.7), quadratically integrable over the set $|z| \geqq \epsilon$.

Our first step will be to replace the function Ω in (2.3) by the development (2.2) and prove the equation

$$\tilde{f}_\epsilon(x) = \sum_{n=1}^{\infty} a_n(x) \tilde{f}_{n,\epsilon}(x), \tag{2.4}$$

where

$$\tilde{f}_{n,\epsilon}(x) = (2\pi)^{-k/2} \int_{|z| \geqq \epsilon} f(y) \frac{Y_n(z')}{|z|^k} dy. \tag{2.5}$$

Let us denote the Nth partial sum of the series (2.1) by $S_N(x, z')$. Since, for fixed x, $S_N(x, z')$ converges to $\Omega(x, z')$ over Σ, in norm L^2, the product $S_N(x, z')|z|^{-k}$ converges, in the same norm, to $\Omega(x, z')|z|^{-k}$ over the set $|z| \geqq \epsilon$. Hence, by Schwarz's inequality,

$$\int_{|z| \geqq \epsilon} f(y) S_N(x, z') |z|^{-k} dy \to \int_{|z| \geqq \epsilon} f(y) \Omega(x, z') |z|^{-k} dy,$$

which is (2.4).

It may be added that the series in (2.4) converges absolutely. On account of (2.2) it is enough to prove the convergence of $\sum |\tilde{f}_{n,\epsilon}(x)|^2$. For this purpose we observe that the integrals (2.5) are the Fourier coefficients of $f(y) = f(x-z)$ with respect to the functions equal to $(2\pi)^{-k/2} Y_n(z')|z|^{-k}$ for $|z| \geqq \epsilon$ and equal to zero elsewhere. The latter functions form an orthogonal system in E^k, on account of the orthogonality of the $Y_n(z')$ over Σ. The norms of these functions are not 1 but are bounded away from zero so that the convergence of $\sum |\tilde{f}_{n,\epsilon}(x)|^2$ is a consequence of Bessel's inequality.

Let us now integrate the inequality (see (2.4))

$$|\tilde{f}_\epsilon(x)|^2 \leqq \left(\sum_1^\infty |a_n(x)|^2 \right) \left(\sum_1^\infty |\tilde{f}_{n,\epsilon}(x)|^2 \right) \leqq A \sum_1^\infty |\tilde{f}_{n,\epsilon}(x)|^2 \tag{2.6}$$

over E^k. We get

$$\|\tilde{f}_\epsilon\|^2 \leqq A \sum_1^\infty \|\tilde{f}_{n,\epsilon}(x)\|^2. \tag{2.7}$$

Suppose we can prove that

$$\|\tilde{f}_{n,\epsilon}\| \leqq \frac{A}{n} \|f\| \qquad (n = 1, 2, \cdots) \tag{2.8}$$

with A independent of n and ϵ. The inequality (1.8) will then follow.

From (2.8) will also follow that

$$\|\tilde{f}_\epsilon - \tilde{f}_{\epsilon'}\| \to 0 \qquad (\epsilon, \epsilon' \to 0)$$

since it is known that for each particular value of n we have

$$\|\tilde{f}_{n,\epsilon} - \tilde{f}_{n,\epsilon'}\| \to 0 \qquad \text{as } \epsilon, \epsilon' \to 0$$

(see [2, p. 89]). Hence there will exist a function $\tilde{f} \in L^2$ such that

$$\|\tilde{f} - \tilde{f}_\epsilon\| \to 0,$$

which in conjunction with (1.8) implies (1.9).

Thus our main problem now is to prove (2.8).

Let us revert to (2.5) and let us denote the Fourier transform of any function $\hat{f}$ by f. In other words,

$$\hat{f}(x) = (2\pi)^{-k/2} \int_{E^k} e^{-i(x,y)} f(y) dy$$

where (x, y) denotes the scalar product $\xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_k\eta_k$ of x and y. The equation (2.5) tells us that $\tilde{f}_\epsilon(x)$ is the convolution (with normalizing factor $(2\pi)^{-k/2}$) of f and of the function $g_{n,\epsilon}(y)$ equal to $Y_n(y')|y|^{-k}$ for $|y| \geqq \epsilon$ and to zero otherwise. Suppose that f is both in L^2 and L. Then $\tilde{f}_{n,\epsilon}$ is of the class L^2 and its Fourier transform is the product of the transforms of f and $g_{n,\epsilon}$:

$$\widehat{\tilde{f}_{n,\epsilon}} = \hat{f}\hat{g}_{n,\epsilon}.$$

Moreover, by the Plancherel-Parseval theorem,

$$\|\tilde{f}_{n,\epsilon}\| = \|\widehat{\tilde{f}_{n,\epsilon}}\| = \|\hat{f}\,\hat{g}_{n,\epsilon}\| \leqq \operatorname*{Sup}_x |\hat{g}_{n,\epsilon}(x)|\,\|\hat{f}\| = \operatorname*{Sup}_x |\hat{g}_{n,\epsilon}(x)|\,\|f\|,$$

and (2.8) will follow, at least for f simultaneously in L and L^2, if we show that

$$|\hat{g}_{n,\epsilon}(x)| \leqq \frac{A}{n}. \tag{2.9}$$

That this will, in turn, imply (2.8) for general f quadratically integrable is immediate. For we may first apply (2.8) to the function $f_R(x)$ which coincides with $f(x)$ for $|x| \leqq R$ and is zero elsewhere, and then making R tend to infinity and applying Fatou's lemma we obtain (2.8) in the general case.

Thus our problem has been reduced to (2.9). The proof of the latter inequality requires two lemmas.

3. Lemma 1. *Let $Y_n(y')$ be a spherical function of order n and suppose that $\|Y_n(y')\| = 1$. Then*

$$|Y_n(y')| \leqq A n^{(k-2)/2}. \tag{3.1}$$

The case $k=2$ being trivial, we may assume that $k \geqq 3$. We shall systematically use the notation

$$\lambda = (k-2)/2 \tag{3.2}$$

and shall denote by P_n^λ the ultraspherical (Gegenbauer) polynomials defined by the equation

$$(1 - 2w\cos\gamma + w^2)^{-\lambda} = \sum_{n=0}^{\infty} w^n P_n^\lambda(\cos\gamma). \tag{3.3}$$

Then, for any x with $|x| = 1$ we have

$$Y_n(x) = \frac{\Gamma(\lambda)(n+\lambda)}{2\pi^{\lambda+1}} \int_\Sigma P_n^\lambda(\cos\gamma) Y_n(y')dy' \tag{3.4}$$

where γ is the angle between the vectors x and y' and the integral is zero if we replace $Y_n(y')$ by $Y_m(y')$, with $m \neq n$ (see [4]). By Schwarz's inequality,

$$\begin{aligned} |Y_n(x)| &\leqq An \left\{ \int_\Sigma [P_n^\lambda(\cos\gamma)]^2 dy' \right\}^{1/2} \left\{ \int_\Sigma |Y_n(y')|^2 dy' \right\}^{1/2} \\ &\leqq An \left\{ \int_\Sigma [P_n^\lambda(\cos\gamma)]^2 dy' \right\}^{1/2} \end{aligned} \tag{3.5}$$

The value of the last integral may be obtained if for Y_n we take the spherical function P_n^λ (cos δ), where δ is the angle of y' with a fixed axis through the origin. Then for x on that axis, $Y_n(x) = P_n^\lambda(1)$, and (3.4) reduces to

$$P_n^\lambda(1) = \frac{\Gamma(\lambda)(n+\lambda)}{2\pi^{\lambda+1}} \int_\Sigma [P_n^\lambda(\cos\gamma)]^2 dy'. \tag{3.6}$$

On the other hand, (3.3) gives

$$\sum_0^\infty P_n^\lambda(1) w^n = (1-w)^{-(k-2)},$$

which shows that $P_n^\lambda(1)$ is exactly of the order n^{k-3} and this in conjunction with (3.5) and (3.6) proves the lemma.

Let us now consider the Fourier transform of $g_{n,\epsilon}$. We have

$$\begin{aligned} \hat{g}_{n,\epsilon}(x) &= (2\pi)^{-k/2} \int_{|y| \geqq \epsilon} e^{i(x,y)} \frac{Y_n(y')}{|y|^k} dy \\ &= (2\pi)^{-k/2} \int_{|y| \geqq \epsilon} e^{ir\rho\cos\gamma} \frac{Y_n(y')}{|y|^k} dy, \end{aligned} \tag{3.7}$$

where

$$r = |x|, \qquad \rho = |y|, \qquad r\rho\cos\gamma = (x, y).$$

The last integral is defined as the limit for $R\to\infty$ of

$$\int_{\epsilon\leq\rho\leq R} e^{ir\rho\cos\gamma}\frac{Y_n(y')}{|y|^k}dy = \int_\epsilon^R\frac{d\rho}{\rho}\int_\Sigma e^{ir\rho\cos\gamma}Y_n(y')dy' \tag{3.8}$$

$$= \int_{\epsilon r}^{Rr}\frac{d\rho}{\rho}\int_\Sigma e^{i\rho\cos\gamma}Y_n(y')dy'.$$

Let us first consider the case $k\geqq 3$ and use the expansion

$$e^{i\rho\cos\gamma} = 2^\lambda\Gamma(\lambda)\sum_{m=0}^\infty(m+\lambda)i^m\frac{J_{m+\lambda}(\rho)}{\rho^\lambda}P_m^\lambda(\cos\gamma)$$

(see [9, p. 368], J_k is here Bessel's function of order k) which converges absolutely and uniformly for ρ remaining within any finite interval. On account of (3.4), the last integral (3.8) reduces, except for a multiplicative constant depending on λ only, to

$$Y_n(x)\int_{\epsilon r}^{\epsilon R}\frac{J_{n+\lambda}(\rho)}{\rho^{1+\lambda}}d\rho\to Y_n(x)\int_{\epsilon r}^\infty\frac{J_{n+\lambda}(\rho)}{\rho^{1+\lambda}}d\rho \qquad (as\ R\to\infty).$$

Since, by Lemma 1, $|Y_n|\leqq An^\lambda$, the inequality (2.9) will follow if we prove the following

LEMMA 2.

$$\left|\int_h^\infty\frac{J_{n+\lambda}(\rho)}{\rho^{1+\lambda}}d\rho\right|\leqq\frac{A}{n^{1+\lambda}}, \qquad \textit{for } 0<h<\infty;\ n=1,2,\cdots. \tag{3.9}$$

Let us revert for a moment to the case $k=2$. We may then set

$$x=re^{i\theta},\qquad y=\rho e^{i\phi},\qquad Y_n(y')=\alpha e^{in\phi}+\bar{\alpha}e^{-in\phi}.$$

The inner integral on the right of (3.8) reduces then to

$$\int_0^{2\pi}e^{ir\rho\cos(\phi-\theta)}(\alpha e^{in\phi}+\bar{\alpha}e^{-in\phi})d\phi = 2\pi J_n(r\rho)i^n(\alpha e^{in\theta}+\bar{\alpha}e^{-in\theta}),$$

and we are again led to the inequality (3.9) with $\lambda=0$, as it should be.

We could not find Lemma 2 in literature—though the formula for the integral (3.9) in the case $h=0$ is classical (see [9, p. 391] or [8, p. 182]). We are therefore forced to give here a proof of it which is a straightforward adaptation of a proof communicated to us by Professor Szegö for the case of $\lambda=0$. This proof is long and possibly could be simplified. We postpone it to the last section of the paper. Taking Lemma 2 temporarily for granted we may consider Theorem 1 as proved.

In view of the complicated character of the proof of Lemma 2 the following remark may be of some interest. Suppose that the function f is differentiable sufficiently many times and vanishes outside a compact set. Then the integral defining $\tilde{f}(x)$ is obviously convergent at every point x. Moreover the relation (2.4) will hold for $\epsilon=0$. For such functions f we have the inequality

$$\|\tilde{f}\| \leqq A\|f\|, \tag{3.10}$$

provided we have (3.9) with $h=0$, a result which, as we have already observed, is well known. Thus the operator $\tilde{f}$ can be defined directly for functions f forming a set dense in L^2 and for these functions we have (3.10) with A independent of f. Such an operator can be extended to the whole L^2 with preservation of (3.10). This kind of extension is systematically used by Mihlin [5; 6]. Of course using this argument we lose the fact that $\tilde{f}_\epsilon$ converges to $\tilde{f}$ in the metric L^2 for every f in L^2, as $\epsilon \to 0$.

Let us finally make two remarks of a chronological type. 1° Developments of the numerator Ω into series of spherical harmonics was already considered by Giraud [3] (see also Mihlin, loc. cit.). 2° The computation of the Fourier transform of the function $g_{n,\epsilon}$ contains implicitly the important fact that the Fourier transform of $Y_n(y')|y|^{-k}$ is, apart from a numerical factor, the harmonics $Y_n(y')$ itself. This fact was not unknown in the case $k=2$ but for higher values of k seems to have been first published by Bochner [1] whose argument is used in our proof. (For $k=3$ the result was also obtained—independently and almost simultaneously—by Professor G. Szegö, who communicated it to us in a letter. His proof was not published.) Bochner sums the Fourier transforms by the method of Abel, but since we know that the transform actually converges [2, p. 89], this point is irrelevant.

4. The convergence of the series in (2.7) follows from the inequality (2.8) and the convergence of the series $\sum n^{-2}$. The latter fact is rather crude and we have here considerable leeway which might conceivably be used to strengthen Theorem 1. Using instead of (2.6) the inequality

$$|\tilde{f}_\epsilon(x)|^2 \leqq \left(\sum_{n=1}^{\infty} |a_n(x)|^2 n^{-1+\delta}\right)\left(\sum_{n=1}^{\infty} |\tilde{f}_{n,\epsilon}(x)|^2 n^{1-\delta}\right),$$

and applying (2.8) we see that the conclusion of Theorem 1 holds if instead of the boundedness of the function $\sum|a_n(x)|^2$—which is equivalent to condition (1.7)—we assume the boundedness of $\sum|a_n(x)|^2 n^{-1+\delta}$. This will lead us to the following result.

THEOREM 2. *The conclusion of Theorem* 1 *holds if condition* (1.7) *is replaced by*

$$\int_\Sigma |\Omega(x, z')|^p dz' < A \tag{4.1}$$

for any

$$p > 2\frac{k-1}{k}. \tag{4.2}$$

Thus for $k=2$ any $p>1$ will do. However, as k increases indefinitely the expression $2(k-1)/k$ tends to 2 and the exponent 2 of Theorem 1 is the best one valid for all k. In the last section of this paper (see Theorem 3 below) we shall show that for no k can we replace the exponent $2(k-1)/k$ of Theorem 2 by a smaller one. On the other hand, we do know (see [2, p. 91]) that in the special case of $\Omega=\Omega(z')$ condition (4.1) can be replaced by a weaker one, namely

$$\int_\Sigma |\Omega(z')| \log^+ |\Omega(z')| \, dz' < \infty \tag{4.3}$$

for all k. We shall also show later that the proofs of Theorems 1 and 2 give slightly more than actually stated in the theorems.

Let us recall some familiar facts about the polynomials P_n^λ (for all this see, for example, [7, pp. 80 sqq.]). We have

$$(1-x^2)^{\lambda-1/2}P_n^\lambda(x) = \frac{(-2)^n}{n!}\frac{\Gamma(n+\lambda)\Gamma(n+2\lambda)}{\Gamma(\lambda)\Gamma(2n+2\lambda)}\frac{d^n}{dx^n}(1-x^2)^{n+\lambda-1/2}. \tag{4.4}$$

Functions $g(x)$ defined on the segment $-1\leqq x\leqq 1$ can be developed into Fourier series

$$g(x) \sim \sum_0^\infty c_n P_n^\lambda(x)$$

where

$$c_n = \frac{2^{2\lambda-1}}{\pi}\Gamma^2(\lambda)\frac{(n+\lambda)\Gamma(n+1)}{\Gamma(n+2\lambda)}\int_{-1}^{+1} g(x)(1-x^2)^{\lambda-1/2}P_n^\lambda(x)dx. \tag{4.5}$$

We shall, in particular, consider the Fourier series of the function

$$g(x) = (1-x)^{-\alpha} \tag{4.6}$$

where the number α is positive and will be fixed presently. Clearly, we must have

$$\alpha < \lambda + 1/2.$$

Using the formulas (4.5) and (4.4) and applying repeated integration by parts we find (the computation is simple and is omitted here) that c_n is exactly of the order $n^{2\alpha-2\lambda}$.

Let us now consider on Σ the function

$$G = (1-\cos\theta)^{-\alpha} \tag{4.7}$$

where θ is the angle with the polar axis. The function G belongs to the class L^q on Σ if and only if the integral

$$\int_0^\pi \frac{\sin^{k-2}\theta}{(1-\cos\theta)^{\alpha q}}\,d\theta$$

is finite, that is, for

$$q < (k-1)/2\alpha. \tag{4.8}$$

The development of G into spherical harmonics is obtained by replacing x in the Fourier series $\sum_0^\infty c_n P_n^\lambda(x)$ by $\cos\theta$, i.e. is

$$\sum c_n P_n^\lambda(\cos\theta), \quad \text{with} \quad |c_n| \sim n^{2\alpha-2\lambda}.$$

Let us now consider any function $F(x')$ integrable on Σ and an arbitrary function $G(x')$ also integrable on Σ which is however a function of the polar angle θ only. The integral

$$H(x') = \frac{1}{|\Sigma|}\int_\Sigma F(y')G(\cos\gamma)dy', \tag{4.9}$$

where γ denotes the angle between the vectors x' and y', is a sort of convolution of the functions F and G; we may call it the *spherical convolution*. It has many properties of the usual convolution in E^k. In particular, the familiar inequality of W. H. Young

$$\|H\|_t \leqq \|F\|_p \|G\|_q \qquad \left(p,\, q,\, t \geqq 1;\, \frac{1}{t} = \frac{1}{p} + \frac{1}{q} - 1\right)$$

remains valid here, with proof unchanged.

If the developments of F and G into series of spherical harmonics are

$$\sum a_n Y_n(x') \quad \text{and} \quad \sum c_n P_n^\lambda(\cos\theta), \tag{4.10}$$

respectively, then the spherical development of H is

$$\frac{2\pi^{\lambda+1}}{|\Sigma|\,\Gamma(\lambda)} \sum_n a_n c_n \frac{1}{n+\lambda} Y_n(x'). \tag{4.11}$$

The formal proof immediately follows from (4.9) and (3.4). This formal proof is perfectly rigorous if one at least of the developments (4.10) converges absolutely and uniformly. In particular, if we denote by $F_r(x')$ the Abel-Poisson means of the spherical development of $F(x')$ $(0 \leqq r < 1)$, the spherical composition of F_r and G is given by the series (4.11) with a_n replaced by $a_n r^n$. But as is easily seen from the representation of F_r as a Poisson integral of F, we have, on Σ,

$$\|F - F_r\|_1 \to 0 \quad \text{as} \quad r \to 1,$$

and so, by Young's inequality with $p = q = 1$, the spherical composition of F_r and G tends in norm L to the spherical composition of F and G, which immediately proves that (4.11) is the development of H into spherical harmonics.

Suppose now that for our G we take the function (4.7). Then the cofactor of $a_n Y_n$ in (4.11) is exactly of the order $n^{2\alpha-2\lambda-1}$. If $2\alpha-2\lambda-1$ exceeds $-1/2$, i.e. if

$$2\alpha > k - 3/2, \tag{4.12}$$

and if the function H is quadratically integrable over Σ, then the series

$$\sum |a_n|^2 n^{-1+\delta} \tag{4.13}$$

converges for some $\delta > 0$. Thus the last series will converge provided $F \in L^p$ and provided we can find a $q > 1$ and a number $\alpha > 0$ such that the conditions (4.8) and $1/2 \geqq p^{-1} + q^{-1} - 1$ are satisfied. It is easily seen that the last two conditions can be satisfied if we have (4.2).

Summarizing, for any function $F(z') \sim \sum a_n Y_n(z')$ on Σ and of the class L^p, with p satisfying (4.2), the series (4.13) converges for some $\delta > 0$. If for $F(z')$ we take $\Omega(x, z')$, with x fixed, and if we consider the development (2.1), then the assumptions (4.1) and (4.2) will imply the uniform boundedness of $\sum |a_n(x)|^2 n^{-1+\delta}$ for some $\delta > 0$ and the proof of Theorem 2 is completed.

In the above proof we implicitly assumed that $k > 2$. For $k = 2$ the proof is analogous if we take $g = (1 - e^{i\theta})^{-\alpha}$. In this case we could also appeal to a well known result of Hardy and Littlewood asserting that if $F \in L^p$, then the fractional integral of order β belongs to the class L^t with t defined by the equation $1/t = 1/p - \beta$, so that again we would have the convergence of (4.13) for some $\delta > 0$ provided $F \in L^p$, $p > 1$. The proof given previously is of course more elementary.

5. **Remarks.** 1° We have mentioned above that from the proofs of Theorems 1 and 2 we can deduce slightly more than actually stated. For let $\epsilon_1, \epsilon_2, \cdots$ be a sequence of positive variables and let us replace on the right the factors $\tilde{f}_{n,\epsilon}$ by $\tilde{f}_{n,\epsilon_n}$. Let the resulting sum be denoted by $\tilde{f}_{\epsilon_1\epsilon_2\cdots\epsilon_n\cdots}$ so that

$$\tilde{f}_{\epsilon_1\epsilon_2\cdots\epsilon_n\cdots} = \sum_1^\infty a_n(x)\tilde{f}_{n,\epsilon_n}(x).$$

Then the proof of Theorem 1 shows that under its assumption we have

$$\|\tilde{f}_{\epsilon_1\epsilon_2\cdots\epsilon_n\cdots}\| \leqq A \tag{5.1}$$

and

$$\|\tilde{f}_{\epsilon_1\epsilon_2\cdots\epsilon_n\cdots} - \tilde{f}\| \to 0 \tag{5.2}$$

if each individual ϵ_n tends to zero.

2° Let us consider for a moment the case $k=2$ and suppose that the function Ω depends on z' only, i.e. is a function of an angle θ, $0\leqq\theta\leqq 2\pi$. In this case the equation (2.1) takes the form

$$\Omega(\theta) \sim \sum_{-\infty}^{+\infty}{}' a_n e^{in\theta},$$

where the a_n are constants independent of x. The inequality (2.8) becomes

$$\|\tilde{f}_{n,\epsilon}\| \leqq \frac{A}{|n|}\|f\| \qquad (n = \pm 1, \pm 2, \cdots).$$

Let us now assume that the function $\Omega(\theta) \log^+ |\Omega(\theta)|$ is integrable over $(0, 2\pi)$. It is known that then the series $\sum' |a_n n^{-1}|$ converges absolutely and

$$\sum_{-\infty}^{+\infty}{}' \left|\frac{a_n}{n}\right| \leqq A\int_0^{2\pi} |\Omega| \log^+ |\Omega| d\theta + A \tag{5.3}$$

(see [10, p. 235, Ex. 5]) and from 2.4—or rather its analogue in the case $k=2$—we obtain the inequalities (1.8) and (1.9).

We obtained these inequalities under the assumption that $\Omega \log^+ |\Omega|$ is integrable over $0\leqq\theta\leqq 2\pi$. Of course, the result is not new, but the present argument shows that the generalizations (5.1) and (5.2) are valid in the case $k=2$ if $\Omega \log^+ |\Omega|$ is integrable. The argument is not extensible to higher values of k since, though the inequality (5.3) is extensible to general uniformly bounded orthonormal systems (loc. cit.), the condition of boundedness is essential here and the orthonormal systems $\{Y_n(z')\}$ we come across when $k\geqq 3$ are no longer uniformly bounded.

6. We shall now prove Lemma 2. As we have observed, the inequality (3.9) is certainly true if $h=0$, so that Lemma 2 is equivalent to the inequality

$$\left|\int_0^h \frac{J_{n+\lambda}(\rho)}{\rho^{1+\lambda}} d\rho\right| \leqq \frac{A}{n^{1+\lambda}} \qquad (n = 1, 2, \cdots; h \geqq 0). \tag{6.1}$$

Let us write $\nu=n+\lambda$. We shall consider four special cases, namely,

1° $0 \leqq h \leqq \nu/2$, 2° $\nu/2 \leqq h \leqq \nu$, 3° $\nu \leqq h \leqq 2\nu$, 4° $h \geqq 2\nu$.

In case 1°, the classical formula (see e.g. [9, p. 48])

$$|J_\nu(\rho)| \leqq \left|\frac{(\rho/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)}\int_{-1}^{+1}(1 - t^2)^{\nu-1/2}e^{i\rho t}dt\right| \leqq \frac{A(\rho/2)^\nu}{(\nu - 1)!}$$

coupled with Stirling's formula for $(\nu-1)!$ shows that the integral in (6.1) is uniformly $\leqq Aq^\nu$, where q is a positive number less than 1, and (6.1) is surely true.

In case 2° we use the formula [9, p. 257, (6)]

$$\int_0^{\nu} J_\nu(\rho)d\rho \simeq \frac{1}{3},$$

and since $J_\nu(\rho)$ is positive for $0 \leqq \gamma \leqq \rho$, we have

$$\int_{\nu/2}^{h} \frac{J_\nu(\rho)}{\rho^{\lambda+1}} d\rho \leqq A\nu^{-\lambda-1} \int_{\nu/2}^{\nu} J_\nu(\rho)d\rho \leqq A\nu^{-\lambda-1} \leqq An^{-\lambda-1},$$

which in conjunction with case 1° again gives (6.1).

In case 4° we use the differential equation of J_ν which may be written

$$\frac{J_\nu(\rho)}{\rho^{\lambda+1}} = -\frac{J_\nu'(\rho)}{\rho^\lambda(\rho^2-\nu^2)} - \frac{J_\nu''(\rho)}{\rho^{\lambda-1}(\rho^2-\nu^2)}.$$

Let us integrate this over $h \leqq \rho < \infty$. Since $|J_\nu'(\rho)| \leqq 1$, the first term on the right is, numerically, $\leqq A\rho^{-2-\lambda}$ and its integral is $\leqq Ah^{-\lambda-1} \leqq A\nu^{-\lambda-1}$. To the integral of the second term we apply the second mean-value theorem and remove the decreasing factor $[\rho^{\lambda-1}(\rho^2-\nu^2)]^{-1}$, which shows again that the integral is $\leqq A\nu^{-\lambda-1}$. This proves (3.9) in case 4°.

It remains to prove (6.1) in case 3°. Let $\nu \leqq h \leqq 2\nu$. The second mean-value theorem gives

$$\int_\nu^h \frac{J_\nu(\rho)}{\rho^{\lambda+1}} d\rho = \nu^{-\lambda-1} \int_\nu^{h'} J_\nu(\rho)d\rho \qquad (\nu < h' < 2\nu)$$

which indicates that it is enough to prove the boundedness of the last integral. We write this condition in the form

$$(6.2) \qquad \int_1^\xi J_\nu(\nu\rho)d\rho = O\left(\frac{1}{\nu}\right) \qquad (1 < \xi < 2).$$

This part of the argument is the least simple. We set $\rho = \sec \beta$, and use Watson's formula [9, p. 252, (5)] valid in the "transitional region":

$$J_\nu(\nu \sec \beta) = (1/3) \tan \beta \cdot \cos \nu B[J_{-1/3}(t) + J_{1/3}(t)] + 3^{-1/2} \tan \beta \cdot \sin \nu B[J_{-1/3}(t) - J_{1/3}(t)] + O(1/\nu),$$

where

$$B = \tan \beta - (1/3) \tan^3 \beta - \beta, \qquad t = (1/3)\nu \tan^3 \beta,$$

and the "O" is an absolute one, provided $0 \leqq \beta \leqq \beta_0$, β_0 being any fixed constant (in our case $\sec \beta_0 = 2$).

If we drop the term $O(1/\nu)$, we obtain an approximate formula for J_ν and the error committed in the integral (6.2) will be $O(1/\nu)$, a quantity unimportant for our purposes.

First let us suppose that $t \leqq 1$ in the interval $1 \leqq \rho \leqq \xi$. Then $J_{-1/3} \pm J_{1/3} = O(t^{-1/3})$ and the left side of (6.2) is, numerically,

$$\leqq \int_{t\leqq 1} \left| J_\nu(\nu \sec \beta) \right| \frac{\sin \beta}{\cos^2 \beta} d\beta \leqq A \int_{t\leqq 1} \tan^2 \beta t^{-1/3} d\beta \leqq A\nu^{-1/3} \int_{t\leqq 1} \tan \beta d\beta$$

$$\leqq A\nu^{-1/3} \int_{t\leqq 1} \beta d\beta \leqq A\nu^{-1/3} \cdot \nu^{-2/3} = A\nu^{-1}.$$

Second, let $t > 1$. Then

$$J_{\mp 1/3}(t) = \left(\frac{2}{\pi t}\right)^{1/2} \cos\left(t \pm \frac{\pi}{6} - \frac{\pi}{4}\right) + O(t^{-3/2}),$$

$$J_{-1/3} + J_{1/3} = 3^{1/2}\left(\frac{2}{\pi t}\right)^{1/2} \cos\left(t - \frac{\pi}{4}\right) + O(t^{-3/2}),$$

$$J_{-1/3} - J_{1/3} = -\left(\frac{2}{\pi t}\right)^{1/2} \sin\left(t - \frac{\pi}{4}\right) + O(t^{-3/2}),$$

$$J_\nu(\nu \sec \beta) = 3^{-1/2} \tan \beta \left(\frac{2}{\pi t}\right)^{1/2} \cos\left(\nu B + t - \frac{\pi}{4}\right) + O(\tan \beta \cdot t^{-3/2})$$

$$= 3^{-1/2} \tan \beta \left(\frac{2}{\pi t}\right)^{1/2} \cos\left[\nu(\tan \beta - \beta) - \frac{\pi}{4}\right] + O(\beta t^{-3/2}).$$

The contribution of the "O" term here to that part of the integral (6.2) which corresponds to $t \geqq 1$ does not exceed

$$A \int_{t\geqq 1,\, \beta\leqq\beta_0} \beta^2 t^{-3/2} dt \leqq A\nu^{-3/2} \int_{t\geqq 1} \beta^{-5/2} d\beta \leqq A\nu^{-3/2} \cdot A\nu^{1/2} = A\nu^{-1},$$

and it remains to consider the integral (in which $\beta_1 \leqq \beta_0$)

$$\int_{t\geqq 1,\, \beta\leqq\beta_1} \tan \beta \cdot t^{-1/2} \cdot \cos\left[\nu(\tan \beta - \beta) - \frac{\pi}{4}\right] \tan \beta \sec \beta d\beta$$

$$= \int A\nu^{-1/2} \tan^{1/2} \beta \sec \beta \cos\left[\nu(\tan \beta - \beta) - \frac{\pi}{4}\right] d\beta.$$

Let us now introduce a new variable $x = \tan \beta - \beta$ which is an increasing function of β. Clearly, for $\beta \to 0$ we have

$$x \simeq \frac{1}{3}\beta^3, \qquad \tan^{1/2} \beta \sec \beta \simeq A x^{1/6} \qquad \frac{d\beta}{dx} \simeq A x^{-2/3}$$

and the ratios of both sides in the last two equivalences are monotone functions. Therefore, applying the second mean-value theorem we see that the last integral is

$$A\nu^{-1/2}\int x^{-1/2}\cos\left(\nu x-\frac{\pi}{4}\right)dx$$

where the integral is extended over some interval of the positive real axis. Making the substitution $\nu x=y$ and observing that $|\int y^{-1/2}\cos(y-\pi/4)dy|\leqq A$, we see that (6.4) is absolutely $A\nu^{-1}$, and the proof of Lemma 2 is complete.

7. THEOREM 3. *If in the inequality* (4.1) *we take* $p=2(k-1)/k$, *the transform* $\tilde{f}$ *of an* $f\in L^2$ *need not be in* L^2.

For let us take for $f(y)$ the function equal to 1 for $|y|\leqq 1$ and equal to zero elsewhere. It belongs to L^2 (to any L^q, $q>0$). Let p be any positive number.

Let us assume that $\Omega(x, z')=0$ for $|x|\leqq 2$. For $|x|>2$ we define $\Omega(x, z')$ as follows:

1° It is equal to $|x|^{(k-1)/r}$ for the points y of each ray from x intersecting Σ;

2° It is equal to $-|x|^{(k-1)/r}$ for the points y of the rays opposite to those in 1°;

3° It is equal to zero on all other rays from x.

Then

$$\int_\Sigma \Omega(x, z')dz' = 0, \qquad \int_\Sigma |\Omega(x, z')|^p dz' < A,$$

It is not difficult to see that for the function f just defined,

$$|\tilde{f}(x)| = \left|\int_\Sigma \Omega(x, z')|z|^{-k}f(y)dy\right| \simeq \frac{C}{|x|^\alpha} \qquad \text{as } |x|\to\infty,$$

where C is a constant and $\alpha=k-(k-1)/p$. If we want $\tilde{f}(x)$ to be in L^2 we must assume that $\alpha>k/2$, or, what is the same thing, $p>2(k-1)/k$. This completes the proof.

If we assume something more about the symmetrical structure of the kernel K, the conclusion of Theorem 2 may be considerably strengthened. To this we shall return in another paper.

Added in Proof. A new and simpler proof of Lemma 2 was found by L. Lorch and P. Szegö and is to appear in Duke Math. J.

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